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THE

MATHEMATICAL GAZETTE.

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MECHANICAL SIMILITUDE.

BY G. GREENHILL.

AFTER Geometrical Similitude has been retained by making two structures, animal, machine, or steamship, from the same drawing but to a different scale, there is still the remaining variable of the material of construction to be taken into account; and a further question arises of the Mechanical Similitude in a calculation of the relative velocity and stress in the parts.

The first statement of the problem is to be found in the *Principia*, Book II, Section VII, Prop. XXXII, Theorem XXVI.

In its application to the steamship the principle goes by the name of *Froude's Law*; it is applied to infer the performance of the full size steamer by calculation on the record of the model in the experimental tank.

An experiment on the scale of the Great Eastern is called a failure, and gives little information for its cost. But the small scale experiment is most instructive when theory is tested to destruction, and at small expense.

Before these experimental tanks were in existence the naval architect was compelled to turn for guidance to the performance of the successful design of full size, and argue by Froude's Law in making his design for a larger scale.

Thus the Campania and Lucania could be taken as a model when the advance was contemplated to the size and speed of the Mauretania and Lusitania.

In this case the steamer is designed for a voyage of given length, the diameter of action with its coal supply; Froude's Law can then be stated in popular language—"One per cent increase in speed, or decrease of chronometer hours of the voyage, requires two per cent increase in length, six per cent increase in tonnage and coal supply, and seven per cent increase in horse-power and machinery crew."

Thus geometrical scale can be preserved in the coal bunkers, but the boilers are out of proportion, and tend to fill up the vessel as speed and size is increased.

Revolutions are then diminished one per cent, but speed of moving parts is increased one per cent, and steam pressure two per cent.

The comparison is stated in a tabular form, with a large and small letter to denote corresponding quantities for full size and model, with the corresponding power or index of f, the speed ratio.

	Full Size.	Model.	Power.	Index.
Speeds in knots	K	k	f	1
Length of voyage, in sea miles	D	D	f^0	0
Chronometer hours of voyage	$\frac{D}{K}$	$\frac{D}{k}$	f^{-1}	−1
Length in feet	X	x	f^2	2
Tonnage displacement	T	t	f^6	6
Resistance in pounds	R	r	f^6	6
,, at 20 lb/ton	$20T$	$20t$	f^6	6
Horse-power	$\frac{RK}{330}=\frac{TK}{16{\cdot}5}$	$\frac{rk}{330}=\frac{tk}{16{\cdot}5}$	f^7	7
Coal in tons	C	c	f^6	6
,, ,, at b lb/H.P. hour	$\frac{b}{2240}\cdot\frac{RD}{330}$	$\frac{b}{2240}\cdot\frac{rD}{330}$	f^6	6
Revolutions	N	n	f^{-1}	−1
Length of stroke	L	l	f^2	2
Piston area	A	a	f^4	4
Steam pressure	P	p	f^2	2
Indicated horse-power	$\frac{PLAN}{33{,}000}$	$\frac{plan}{33{,}000}$	f^7	7
Horse-power per ton	$\frac{K}{16{\cdot}5}$	$\frac{k}{16{\cdot}5}$	f^1	1

The translation of the power of f and its index into a percentage change is in accordance with the statement in the language of the practical man, that one % increase in f will give q % increase in f^q, as the equivalent of the fundamental theorem in the Differential Calculus, that qf^{q-1} is the differential coefficient of f^q with respect to f.

In Froude's statement, incorporated in the Table, the resistance, R, r, at the corresponding speed, K, k, is taken as proportional to the displacement tonnage, T, t; so that full size and model experience the same resistance estimated in lb/ton, $R/T=r/t$, as on a railway, equivalent to an incline resistance of one in $2240(T/R=t/r)$.

An average resistance has been estimated at 20 lb/ton, an equivalent incline of one in 112, so that steamship and railroad resistance are not very different.

If it is assumed otherwise that the resistance can be divided into two factors, one of them the wetted surface, S, s, proportional to X^2, x^2, or K^4, k^4, the other factor of the resistance is the square of the velocity, K^2, k^2.

This is for the frictional drag under water, according to the law adopted usually of wet friction; but the wave making resistance is to be added, assumed proportional to the displacement.

A nautical mile of 1000 fathom is taken here as 6000 feet, making the knot a speed of 100 feet a minute, and one H.P. 330 knots-pounds.

At 20 lb/ton, and $R=20T$, the H.P./ton $=\frac{20K}{330}=\frac{K}{16{\cdot}5}$; and at this rate a 23·5 knot steamer of 26,000 tons would require H.P. 37,000, and 4200 tons of coal for 3000 miles.

The coal/ton-mile $=\frac{b}{2240}\frac{\text{H.P.}}{KT}=\frac{20b}{2240\times 330}$, is the same for all; and the coal/ton over a voyage of D miles is $\frac{20bD}{2240\times 330}=\frac{20\times 2\times 3000}{2240\times 330}$, say $\frac{1}{6}$, for $D=3000$, estimating b at 2 lb/H.P. hour, a liberal estimate, which can be halved with oil fuel.

Selecting then as a model a successful Atlantic steamer, Campania or Lucania of 13,000 tons, performing the voyage of 3000 miles at 20 knots in 150 hours, with 19,000 H.P., a calculation of the design on Froude's Law for a speed of 25 knots would give 50,000 tons and 90,000 H.P.; and the coal supply would be raised from 2500 to 9000 tons about. But these dimensions are too great for the entry of the port at each end.

To obtain a sustained sea speed of 25 knots 120 hours, on a smaller tonnage, the fraction of space devoted to coal and machinery must be increased.

Beginning with a design where the tonnage was increased as the cube of the speed, instead of the sixth power, as on Froude's Law, from 13,000 to 25,000 tons, and the H.P. as the fifth power, from 19,000 to 58,000, the H.P./ton must vary then as the square of the speed, rising from 1·46 to 2·28.

But as this design would sacrifice nearly all the interior space to machinery and coal, a compromise was made in the Mauretania-Lusitania design, 42,000 tons with about 70,000 H.P. at 25 knots, and these were not too large for the terminal ports.

In fluid friction it is usual to assume for calculation that the frictional drag over a surface is a fraction e of the pressure $\rho\frac{V^2}{g}$, lb/ft², of the normal impact as dust, of density ρ lb/ft³, with velocity V f/s; so that over S ft², the resistance

$$R=Se\rho\frac{V^2}{g}, \text{ lb};$$

and thence an estimate of e can be inferred when the wetted surface S is measured of the steamer; but e varies considerably with the time out of dock.

An average value $e=\frac{1}{400}$ may be taken, and this is applicable also to the frictional drag on an airship.

In air of average density, bulking 12·5 ft³/lb, and taking $g=32$, and at a velocity V f/s, or K knots,

$$\frac{R}{S}=e\rho\frac{V^2}{g}=\frac{V^2}{400\times 12{\cdot}5\times 32}=\left(\frac{V}{400}\right)^2=\left(\frac{K}{240}\right)^2, \text{ lb/ft}^2;$$

and the thrust-horse-power (T.H.P.) required for the propulsion of the airship is

$$\text{T.H.P.}=\frac{RV}{550}=\frac{RK}{330}=\frac{SV^3}{88{,}000{,}000}=\frac{SV^3}{88 \text{ million}}=\frac{SK^3}{19 \text{ million}}.$$

An airship body, 500 feet long and 55 in diameter, with pointed ends, will have $S=88{,}000$ ft² about, requiring $\text{T.H.P.}=\frac{V^3}{1000}$ at V f/s.

Thus at 60 miles an hour, $V=88$, the T.H.P. would be 680.

But the indicated horse-power (I.H.P.) would require to be raised to about 960, at 70% efficiency, and at an estimate of 1 lb/I.H.P. hour, the petrol required for a flight of 12 hours would be 12,000 lb, nearly 6 tons, for a radius of action, there and back, of 360 miles.

The displacement of the body of the airship will be nearly

$$\tfrac{1}{4}\pi(55)^2\times 500=1{,}200{,}000 \text{ ft}^3,$$

and this is about 42 tons of air, bulking $12{\cdot}5\times 2240=28{,}000$ ft³/ton, giving a net lift of say 30 tons when filled with hydrogen gas.

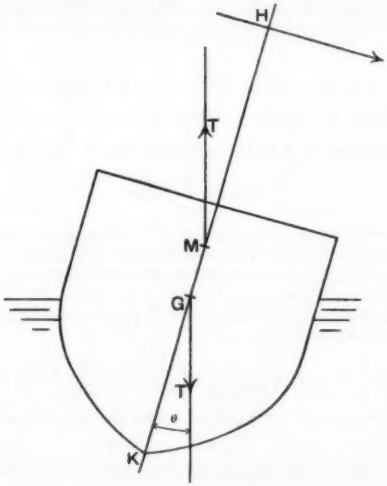
Allowing 20 tons tare for structure and machinery, there remains 10 tons for crew and provisions, petrol 6 tons for a flight of 12 hours, and a supply of bombs, say twenty 100 kg bombs, weighing 2 tons.

At 50-knot speed maintained for 60 hours, a diameter (not radius) of action of 3000 miles, the petrol required would be over 22 tons, more than the airship could carry.

We are promised an airship, 300 metres long and diameter 30 m, displacing over 250 tons of air, and with I.H.P. 2700 at 90 km/hour, the radius of action could be made 2700 km, a flight of 60 hours, or a diameter of action of 5400 km or 2900 sea miles; on a consumption of 80 tons of petrol, estimated at 0·5 kg/I.H.P. hour.

Similar statements will hold in comparing the performance of the model with the full size sailing ship: here in the comparison the wind on the model and its speed are reduced as the square root of the length or sixth root of the tonnage, for the model to behave properly; and the mortification of the youthful model maker is consoled when he finds his little ship will not behave properly unless ballasted to an extent out of all proportion.

Assume for a numerical contrast, $f=10$, $K=10k$, $X=100x$, T/t a million.



With air bulking $C=12{\cdot}5$, ft³/lb, we take the pressure of the wind, at W f/s or Q knots,

$$\frac{W^2}{Cg}=\left(\frac{W}{20}\right)^2=\left(\frac{Q}{12}\right)^2, \text{ lb/ft}^2;$$

and on the geometrical scale, with ship and model heeled over to the same angle θ,

Sail area, ft², - - - - - -	A	a	f^4
Heeling moment of the wind, acting at a height, H, h, in ft-lb, - -	$HA\left(\frac{W}{20}\right)^2$	$ha\left(\frac{w}{20}\right)^2$	f^5
Righting moment of the water, ft-lb, -	$2240\ T.GM\sin\theta$	$2240\ t.gm\sin\theta$	f^5

$$\sin\theta=\frac{HA\left(\frac{W}{20}\right)^2}{2240\,T.GM}=\frac{ha\left(\frac{w}{20}\right)^2}{2240\,t.gm},$$

$$\left(\frac{W}{w}\right)^2=\frac{T}{t}\cdot\frac{h}{H}\cdot\frac{a}{A}\cdot\frac{GM}{gm}=\frac{X^3}{x^3}\cdot\frac{x}{X}\cdot\frac{x^2}{X^2}\cdot\frac{X}{x}=\frac{X}{x},$$

as in Froude's Law.

Similar considerations of Mechanical Similitude have been investigated by various writers in respect to animals in the scale of creation (Borelli, 1680; Herbert Spencer), taking all animals as actuated by muscular tissue of the same structure, supposed to have the same resilience, estimated in ft-lb of energy per ft^3.

Thus treating a man's leg as swinging like a pendulum through a fixed definite angle, the length of a pace is proportional to his height, but the number of paces a minute is inversely as the square root of the height; so that his pace of walking and getting over the ground will vary as the square root of his height, as in Froude's Law again.

Mechanical Similitude can also be seen illustrated everywhere in statical structures, such as a building, tower, bridge, or tree.

And from experiments on a small scale with bullets fired through clay, it is possible to infer on the Laws of Mechanical Similitude the behaviour of an artillery projectile in penetrating steel or earth.

THE REPORT OF THE COMMITTEE ON THE TEACHING OF ARITHMETIC IN PUBLIC SCHOOLS.

1. In previous reports issued by the Association, general reforms have been suggested which apply to the teaching of Arithmetic as well as of other subjects, and some detailed suggestions have been made which apply exclusively to Arithmetic. The present report deals at greater length with Arithmetic only—its relation to other subjects, its scope and its treatment.

2. There are certain parts of the subject-matter of the Arithmetic Course that are regarded by many teachers as of little educational value to the ordinary student. The Committee suggest that these should not be considered a necessary part of the Mathematical Course. This suggestion in no way interferes with a teacher who, either from personal preference or local considerations, wishes to take a more detailed course; at the same time, it allows others to omit certain parts of a more specialised character. The Committee hope that in the course of time the parts omitted will cease to be required by examining bodies. The omissions will be discussed in detail below.

3. The Committee recommend that the time set free by the elimination of a part of the usual subject-matter should be spent, partly in securing a higher standard of accuracy in straightforward computation and partly in extending the range of the ordinary school course in Mathematics.

The directions in which this course may be extended are discussed in detail in a recent report issued by the Association entitled "A General Mathematical Syllabus for Non-Specialists in Public Schools."

I. GENERAL CONSIDERATIONS.

4. Throughout the report a distinction is made between (1) straightforward computation, such as the addition of sums of money, the multiplication of decimal fractions, the reduction of a compound to a simple quantity, etc., and (2) applications to more general topics, such as variation, mensuration, finance, etc., of such a nature that questions connected with them may to some extent be regarded as problems. In this connection the Committee make the following suggestions:

(*a*) Although no hard and fast line can be drawn between these two types, the distinction is a useful one. In the first, uniformity of method is desirable and numerical accuracy is all-important. In the second, variety of method is to be encouraged; difficulties arising from mere computation should be avoided at first, the value of the question depending chiefly on the amount of thought that it demands from the student.

(*b*) There is much to be said for a thorough exercise in pure computation. The addition of a dozen numbers, each of five or six digits, or of long sums of money, such as occur in a pass-book, provides a purely mechanical training which is of real value. In such work large numbers will occur naturally, and there is no objection to their use. But in the application of Arithmetic to a variety of subjects, merging into the problem category, the introduction of large numbers is, as a rule, to be deprecated.

(*c*) In simple straightforward questions it is desirable that boys should be taught to check their work carefully so as to secure a high standard of accuracy. For example, in addition they should be able to secure this by adding down the columns as well as up, in multiplication by subsequent division, and so on. In this way, with frequent practice, it should be possible, with reasonable allowance for human imperfections, to rely on boys to obtain an accurate result.

(*d*) In all work of the nature of a problem, the argument should be set out clearly and in detail. Slipshod expressions, such as "96 has cost price 100" or "one workman $= \frac{1}{4}$ of a day," should not be permitted. In assigning marks for such work, it is desirable that a considerable proportion should be given for a clear statement of the argument, and should be withheld when this is absent, even if a correct answer is obtained.

(*e*) In work of the nature of a problem, the argument should, as a rule, be completed as far as possible before any calculation is attempted. This will often lead naturally to the use of brackets.

The Committee make the following further suggestions:

5. Ideas and principles should be made as real as possible by concrete illustrations; *e.g.* the early work on decimal fractions should be combined with weighing and measuring.

6. All questions should be of a kind that may be expected to occur in practice, except in a few cases in which artificial examples are designed to illustrate some special rule or principle. Moreover, the data should always be reasonable, if only to train the student to notice how far the result he obtains agrees with common sense. It will also train him in the later stages of his course to consider the degree of accuracy which may be expected under the conditions imposed.

7. In a great deal of the work, especially where the questions are to some extent of a problem type, the examples should be sufficiently varied. Otherwise there is danger that a pupil may solve them by an imitative process without understanding the meaning of his work. This recommendation does not apply to practice in straightforward computation.

8. It is desirable that boys should have systematic practice in rough approximation, and that they should receive credit for applying rough checks to test their work. In many cases a rough estimate might usefully *precede* the detailed work of an arithmetical computation.

9. The value of mental arithmetic in securing quickness and accuracy is now generally recognised; so also is the value of oral work when a new subject is being introduced. This latter will often be in advance of the written work; *e.g.* simple *viva-voce* examples on ratio or percentage may be taken during a systematic course on the simplification of fractions or the division of decimals.

II. THE SYLLABUS IN DETAIL.

The Committee make the following detailed suggestions connected with the teaching of the separate parts of the course. The order in which these suggestions are given is not necessarily that in which the subjects are taught; *e.g.* the fundamental propositions dealing with the areas and volumes of rectangular figures of integral dimensions will probably precede vulgar fractions, and the multiplication and division of vulgar fractions may be taught partly by examples connected with areas.

A. Four Rules.

10. Subtraction should be taught by the method of complementary addition.

11. Multiplication should begin with the left-hand figure of the multiplier, and the work might be arranged in either of the following ways:

Either		or	
	1763		1763
	341		341
	5289		5289
	7052		7052
	1763		1763
	601183		601183

B. Compound Quantities.

12. Only the following tables need be memorised:

Pounds, shillings, pence, farthings; dollars, cents; francs, centimes; marks, pfennigs.
Days, hours, minutes, seconds.
Miles, furlongs, chains, yards, feet, inches.
Acres, roods, poles; acres, square chains, square yards; cubic yards, cubic feet, cubic inches.
Gallons, quarts, pints.
Tons, hundredweights, quarters, stones, pounds, ounces.
Metre, gram, litre, with the prefixes in common use.

In addition to these tables a certain amount of miscellaneous information should be given, such as the relation between days and years;

links and chains; acres and square miles; bushels, pecks and gallons; the meaning of such terms as gross, score, guinea, crown, florin, etc. The pupil should be told that by statute a gallon of water under certain conditions weighs 10 lb.; and that the weight of a cubic centimetre of water may be taken to be 1 gram.

13. Compound quantities should rarely be given in terms of more than three units.

14. Before going on to Fractions a number of simple problems should be worked, including easy questions of a type usually associated with Unitary Method.

15. Some of the oral work at this stage should be designed to introduce the pupil, by a generalisation of ideas, to the notation of algebra; *e.g.* $£x = 20x$ shillings, a ft. b in. $= 12a + b$ in., etc.

C. Decimal Fractions.

16. There are distinct advantages in taking Decimal Fractions before a full course of Vulgar Fractions, as the former arise naturally as an extension of the decimal notation. But, on the other hand, it will probably be found desirable to take, before finishing a full course of Decimal Fractions, a slight course of Vulgar Fractions, such as will enable the pupil to work easy questions on Unitary Method.

17. Decimal Fractions should be introduced by a large amount of *viva-voce* work; there should also be preliminary work with a graduated ruler, diagrams, squared paper, etc.; and actual measurements and weighing, in metric units, should be undertaken.

18. It is desirable that there should be, in the Junior part at any rate of the Public Schools, a generally accepted method of multiplying and dividing decimal fractions. If this is impossible, in each school at least a standard method should be adopted. And this method should be such that a boy can pass from the multiplication and division of integers to that of decimal fractions, and afterwards, if necessary, to contracted work, with as little disturbance as possible.

19. The position of the decimal point should be checked by a rough approximation.

The following examples illustrate the methods that are suggested for general adoption.

MULTIPLICATION.

To multiply 17·63 by 34·1.

Reduce the multiplier to standard form:

$$17{\cdot}63 \times 34{\cdot}1 = 176{\cdot}3 \times 3{\cdot}41.$$

The process can then be set out *in either of the following ways*:

(*a*)	176·3	(*b*)	176·3
	3·41		3·41
	528·9		528·9
	70·52		70·52
	1·763		1·763
	601·183		601·183

In (*a*) the unit figure of the multiplier is under the right-hand figure of the multiplicand, and the decimal point in the answer under the decimal point in the multiplicand.

In (*b*) the unit figure of the multiplier is under the unit figure of the multiplicand, and all decimal points are in a vertical line.

DIVISION.

To divide 17·63 by 34·1.
Reduce the divisor to standard form :

$$\frac{17{\cdot}63}{34{\cdot}1}=\frac{1{\cdot}763}{3{\cdot}41}.$$

$$\begin{array}{r|l} & \cdot 517\ldots \\ \hline 3{\cdot}41 & 1{\cdot}763 \\ & 1{\cdot}705 \\ \hline & 580 \\ & 341 \\ \hline & 2390 \end{array}$$

The position of the first figure of the quotient is found by inspection, as in short division.

20. Later on it may be desirable to introduce the practice of multiplying and dividing without consideration of the decimal point, the latter being placed by a rough approximation.

21. No rule for the conversion of recurring decimals into fractions need be taught.

D. Metric System.

22. This should be confined (i) to simple examples, such as may be reasonably expected to occur in everyday life in countries which use the metric system, (ii) to such work as is required in connection with scientific studies.

23. Quantities should, as a rule, be expressed in *one*, or at most *two*, units ; *e.g.* 2·75 fr. or 2 fr. 75 c., 3·50 m. or 3 m. 50 cm., 1·240 kg. or 1 kg. 240 gm. Such expressions as 3 Km. 6 Hm. 2 Dm. 5 m. should not be used, except to explain the decimal system of numeration.

24. Pupils should have some idea of the relation of metric units to the British system of units. Metre and yard measures, cubic centimetre, cubic decimetre and cubic inch blocks, litre and quart measures should be accessible.

E. H.C.F. and L.C.M.

25. The method of prime factors should be used, and the numbers taken should not be large.

26. The principle that any common factor of a and b is a factor of $a \sim nb$ (where a, n, b are integers) is useful, but the method of finding H.C.F. by alternate division should not be regarded as essential, and should certainly be postponed.

27. The tests of divisibility are often useful in factorising numbers.

28. The term H.C.F. should be used where numbers are concerned, as G.C.M. refers to quantities. Thus, 2 is a *factor* of 10 and 12, but 2 ft. is a *measure* of 10 ft. and 12 ft.

F. Vulgar Fractions.

29. There should be preliminary work with a graduated ruler, diagrams, squared paper, etc.

30. In explaining the rules it is essential that they should be illustrated by reference to particular cases. A great deal of the work can

be based on the fundamental rule that the value of a fraction is unaltered when the numerator and denominator are multiplied by the same number; the importance of this rule should be emphasised.

31. The numbers involved should not be large. In operating with fractions that have large denominators it is generally advisable to express them as decimals.

32. In order that boys may be able to think in fractions, there should be frequent practice in easy problems such as the following:

What fraction is 3 of 5? Show without working that $3\frac{1}{4} \div \frac{5}{8}$ cannot be $2\frac{9}{28}$. What fraction is $\frac{3}{70}$ of $\frac{36}{7}$? Show that to get $\frac{5}{6}$ of 327 you can subtract $\frac{1}{6}$ of 327 from 327. If 11 copies of a book cost a certain sum, what fraction of that sum will 7 copies cost?

33. Complex fractions should be confined to such types as may reasonably be expected to occur in actual practice; *e.g.* $\frac{13\frac{7}{12}}{20}$ gives 13s. 7d. as a fraction of a pound, $\frac{16\frac{1}{2}}{\frac{2}{3}-\frac{1}{4}}$ shillings may occur in a problem, etc.

34. When there is any ambiguity as to the meaning of a fractional expression, brackets should be used. Thus the distinction between $3\frac{1}{2} \div 2\frac{1}{3} \times 1\frac{1}{2}$ and $3\frac{1}{2} \div 2\frac{1}{3}$ of $1\frac{1}{2}$ introduces a purposeless difficulty; the latter should be written $3\frac{1}{2} \div (2\frac{1}{3} \times 1\frac{1}{2})$. The only convention that should be required is that which governs the interpretation of such expressions as $10\frac{1}{2} - 2\frac{1}{4} \times 1\frac{1}{3} + 3\frac{5}{6}$. This same convention is used in Algebra; *e.g.* in $a - bc + d$.

G. Unitary Method.

35. This need not be a long course, as the easier questions will have been given as problems in Sections A, B; and questions on direct and inverse proportion will be treated more scientifically at a later stage (see paragraphs 55 and 56).

36. Even at this stage there should be some discussion as to how far direct proportion is applicable; *e.g.* it might be pointed out that though at present (1916), by parcel post, 1 lb. costs 4d., 2 lb. costs 5d., and that the cost of a ton of sugar is not, as a rule, 2240 times that of 1 lb. Questions should not be set as examples of Simple Proportion if the assumption that the law of Simple Proportion holds with reasonable accuracy is open to doubt.

37. It is hoped that Compound Proportion questions of an elaborate type will disappear as such. *E.g.* such a question as the following should be avoided as an example of Unitary Method: "If 20 men, working 8 hours a day, dig a trench 300 ft. long, 4 ft. wide and 3 ft. deep, in 4 days, how long, etc." Such a question should not be set until the pupils have reached a stage at which they can write down the answer, in a fractional form, without intermediate steps.

38. The following type of method, introducing a temporary compound unit such as a ton-mile or horse-power-hour, occurs in actual practice, and is recommended by some teachers.

If an engine of 200 horse-power consumes 5 tons of coal in 30 hours, how much will one of 300 horse-power require in 24 hours?

Since the amount of coal required for 200×30 horse-power-hours is 5 tons, the amount of coal required for 300×24 horse-power-hours is $5 \times \frac{300 \times 24}{200 \times 30}$ tons.

H. Square Root.

39. The process of finding a square root should be taught, though it need not necessarily be explained, but the use of square-root tables should be permitted.

40. It is important to explain that numbers, all of which have the same significant figures, have for their square roots one of two sets of significant figures; *e.g.* the square roots of 0·614, 61·4, 6140 are 0·784, 7·84, 78·4, but the square roots of 0·0614, 6·14, 614 are 0·248, 2·48, 24·8.

I. Areas and Volumes (Rectangular Figures).

41. The fundamental propositions (giving the area and volume when the dimensions are known) should be explained, and illustrated by means of diagrams and squared paper, blocks, folded paper, etc.

42. The numerical work involved in the early questions should be of a simple character. For this reason the use of the metric system in preference to the English system at first is recommended.

43. It is recommended that here, as elsewhere, the examples should not be divided unduly into types.

J. Mensuration.

44. The study of mensuration should not be limited to rectangular figures, but should include such figures as the triangle, trapezium, quadrilateral, circle, prism, cylinder, pyramid, cone, sphere.

45. Boys should be familiar at an early stage with the principle "cross-section × length = volume."

46. The formulae for the area and circumference of a circle should be known.

47. Many of the formulae must be accepted by the pupil without proof. But some of these formulae may be checked by rough approximations; *e.g.* by inscribing a regular hexagon in a circle, the student sees that the circumference $>6r$, and, by circumscribing a square about it, that the circumference $<8r$.

48. Emphasis should be laid on the idea of degree in formulae which are given; *e.g.* the student might be asked whether $4\pi r$, $4\pi r^2$, or $4\pi r^3$ is a possible formula for the surface of a sphere of radius r.

49. *Oral* work may be usefully employed at this stage to lead up to the idea of variation.

50. Students should understand that, apart from any explicit instructions, results should be given only to that degree of accuracy which the data seem to justify. In elementary mensuration the student will, as a rule, measure to three significant figures, and should at an early stage be brought to realise that no answer should be given to more than three significant figures. The subject of approximation is discussed in further detail in Section O.

K. Ratio, Percentage, Proportion and Variation.

51. The fundamental ideas of ratio and percentage should be enforced by a considerable amount of oral work. The unmathematical mind in comparing two numbers tends to examine their difference. It is not usually realised, without a definite mathematical training, that in many cases the true test is supplied by their ratio.

52. The idea of percentage is important in view of the common use of percentages in practical affairs. A wide variety of questions, neither elaborate nor difficult, should be given.

53. The tendency to specialise on complicated questions on Profit and Loss is undesirable.

54. When a geometrical result is obtained by drawing and by calculation, the pupil should usually make a rough estimate of the percentage error.

55. There should be, at this stage, a discussion of the law of direct proportion and an investigation of related quantities for which this law may hold (see paragraph 36). It is an advantage if such numbers are obtained experimentally, *e.g.* by measuring the circumference and diameters of various circular cylinders. Such sets of numbers should be tested both arithmetically and by plotting them on squared paper.

56. Inverse proportion can be treated in a similar manner, and then other forms of variation. This part of the subject should not go beyond such forms as $y \propto x^n$, $y=ax+b$, and simple cases of joint variation: *e.g.* Vol. of a cylinder varies jointly as its height and the square of the radius of its base; Vol. of a gas varies inversely as its pressure and directly as its absolute temperature.

57. Whether Variation is considered to belong to Arithmetic or Algebra, its treatment should be largely numerical.

L. Logarithms.

58. Whether this subject is considered to belong to Arithmetic or Algebra, the power to use logarithms intelligently has not been acquired unless the pupil has a clear conception of the functional connection between a number and its logarithm, which may be gained in various elementary ways. One such way is to express numbers as powers of 10.

59. Calculations should be so arranged that a clear distinction is made between numbers and their logarithms.

M. Computation.

60. Owing to the complication of the English system of Weights and Measures, questions are bound to arise which, if solved by unsuitable methods, lead to long and difficult work. Such questions as the following, for example, may easily occur in actual practice:

(*a*) Simple Interest on £63,750 for 22 days at $4\frac{1}{8}$ per cent.

(*b*) The moon makes a complete revolution among the stars in 27 days 7 hrs. 43 min. 11·54 secs.; find the angle through which it turns in a day correct to $\frac{1}{100}''$.

(*c*) Find the exchange value of £873 14s. 6d., taking £1 = 25 fr. 22.

The most difficult part of the question is the choice of method. Opportunities should, therefore, be taken to discuss the suitability of different methods for a given type of question. For example, it could be shown that for multiplying a compound quantity by a large number, the method of Practice is the most suitable; for dividing a sum of money by a large number, to the nearest penny, it is best to decimalise the sum; etc.

Such examples as those above are given as extreme cases. It is not desirable that they should occur in ordinary work, but, as they cannot always be avoided, the student should know how to deal with them.

N. Financial Arithmetic.

61. This comprises Simple and Compound Interest, Present Worth and Discount, Stocks and Shares, and a somewhat miscellaneous group of questions which are typified by Equation of Payments, Rates of Exchange, Bankruptcy, etc.

62. As a considerable body of teachers wishes to treat some of these in very slight detail and omit the rest entirely, it is desirable that the course which is to be regarded as compulsory for all should be of a light character. Those teachers who wish to treat such subjects more fully will still be at liberty to do so.

63. Only the following should be expected from all pupils :

(*a*) Simple Interest (Direct and Inverse).

(*b*) Compound Interest (Direct only).

(*c*) Stocks and Shares of a very simple character.

No elaborate questions, such as those dealing with half-yearly increments, etc., should be given on Compound Interest.

64. The educational importance of Compound Interest lies in the value of the idea of a "Compound Interest Law"; and this is best illustrated from simple examples. The difference between Simple and Compound Interest can with advantage be represented graphically. In actual practice, the work is always done from Compound Interest Tables.

65. The subject of Stocks and Shares should be postponed to a late stage in the school course. The difficulty of the subject arises from unfamiliarity with the procedure of the Stock Exchange; and if, on the ground of utility in after-life, it is desirable that pupils should understand the methods of buying and selling shares, and should realise the exact nature of this commodity, much valuable time will be saved by introducing them to the subject at a period when, being more conversant with everyday affairs, they will more readily understand the ideas involved in and remember the details connected with such transactions.

66. There are few branches of Arithmetic which depend so much on the manner in which they are presented by the teacher as Stocks and Shares. If this subject is treated rather as an element of a course on Civics than as a chapter in Arithmetic, it may be made to serve a real and valuable purpose, although possibly it will be less effective in examination work.

67. Present Worth and Discount, as artificially treated in many text-books, should not be required, and financial problems of a specialised nature should be omitted.

O. Approximation.

68. This is a difficult subject to teach, but one of great importance, as it is most necessary that pupils should realise that all observed values are of an approximate nature.

69. Since results depending on approximate data should not be given to more figures than the data justify, it is desirable that pupils should have some rough rules enabling them to decide how far they may carry their answers.

70. The meaning should be explained of such terms as: correct to the nearest tenth; correct within a tenth; correct to the nearest penny; correct within a penny; correct to 3 places of decimals; significant figures; correct to 3 significant figures; absolute, relative and percentage errors.

71. The following principles are important. There is a diversity of opinion as to how far the average boy can appreciate them, but it would be an advantage if to some extent they could be explained and illustrated.

(*a*) In addition and subtraction the number of decimal places is important. If each given number is correct to n decimal places, the nth decimal place of the answer is only approximately correct; if the given numbers are not all correct to the same number of decimal places, and the one with the greatest absolute error is correct to n decimal places, the nth decimal place of the answer is only approximately correct.

(*b*) In multiplication and division the number of significant figures is important. If each given number is correct to n significant figures, the nth significant figure of the answer is only approximately correct; if the given numbers are not all correct to the same number of significant figures, and the one with the greatest relative error is correct to n significant figures, the nth significant figure of the answer is only approximately correct.

(*c*) In addition and subtraction the maximum *absolute* error in the answer is the sum of the maximum *absolute* errors in the given quantities, though where several quantities are added it is improbable that it will be nearly so great.

(*d*) In multiplication and division, the maximum *relative* error in the answer is approximately the sum of the *relative* errors in the given factors, where these errors are not large, though where several quantities are multiplied or divided it is improbable that it will be nearly so great.

By these means the pupil will be able to determine the number of figures that he should give in his answer, when the data which he is using are given to a known degree of approximation.

72. To state the maximum error is often a better indication of the degree of accuracy to which a result may be given than to state the result to a certain limited number of figures; *e.g.* $27{\cdot}3 \pm 0{\cdot}2$ c.c. is better than 27 c.c. or 27·3 c.c. It should be pointed out that, as a rule, only a very rough statement can be made of the possible error in a result obtained from actual measurements, and that it is generally sufficient to give the error to one significant figure. For example, in measuring a length with an ordinary centimetre rule, the error ought to be not greater than 0·5 mm., and will most probably be greater than 0·1 mm.; it is often difficult to define such errors more accurately.

The last figure of an approximate answer is often indeterminate unless the working is carried to a greater degree of accuracy than is indicated by the question; *e.g.* if an answer, required to the nearest penny, is actually 2s. 4·501d., we cannot determine whether 2s. 4d. or 2s. 5d. is the more accurate without working to three or four decimal places (in pence). In such a case either answer should be accepted. The difficulty can be avoided by asking for the answer "correct within a penny" instead of "to the nearest penny," and other cases may be treated in a similar manner.

P. Problems.

73. The tendency to standardise problems for examination purposes has for a long time been a serious matter.

It is probably true that if certain kinds of problems are to be taught at all, they must be taught systematically much in the same way as other subjects, and illustrated by groups of examples divided into types. This careful consideration of special kinds of problems should be confined to such as involve an important principle or are of considerable intrinsic interest; for the study of a long classified list of problems occupies much time that might be better spent, and is apt to degenerate into unprofitable work for examinations. Problems on relative velocity (*e.g.* on trains passing each other) and inverse questions (*e.g.* those usually described as problems on work) will probably be regarded as of sufficient value to be worth teaching.

On the other hand, the really valuable questions, regarded as problems, are those which are given without a preliminary study of other questions of the same type. These will necessarily be somewhat simpler in character than those described in the previous paragraph, and should be as varied as possible.

The power to deal with such easy miscellaneous problems is an important test of some of the qualities that should be developed by a successful Arithmetic Course; such work should test a boy's power of clear statement and orderly arrangement as well as of mathematical ability.

It is important that it should be recognised both by teachers and examiners that to be of real value such problems should not be classified and divided into groups or studied as standard problems. Problems which have been thus classified lose much of their value as tests of ability, and should be set sparingly by examining bodies.

III. RECAPITULATION OF DETAILED COURSE.

74. The Committee recommend that the syllabus for the Compulsory Arithmetic paper in a pass examination be limited as follows. The numbers refer to the sections of this report.

(1) Four rules, Simple and Compound.

(2) Weights and Measures, as limited in 12.

(3) Decimal Fractions.

(4) Metric System.

(5) H.C.F. and L.C.M. of simple numbers by prime factors.

(6) Vulgar Fractions, as limited in 31, 33.

(7) Unitary Method (see 37).

(8) Square Root.

(9) Areas and Volumes.

(10) Ratio (including Percentage), Proportion and Variation (see 56).

(11) Simple Interest.

(12) Direct Compound Interest, as limited in 63.

(13) Stocks and Shares of a very simple character.

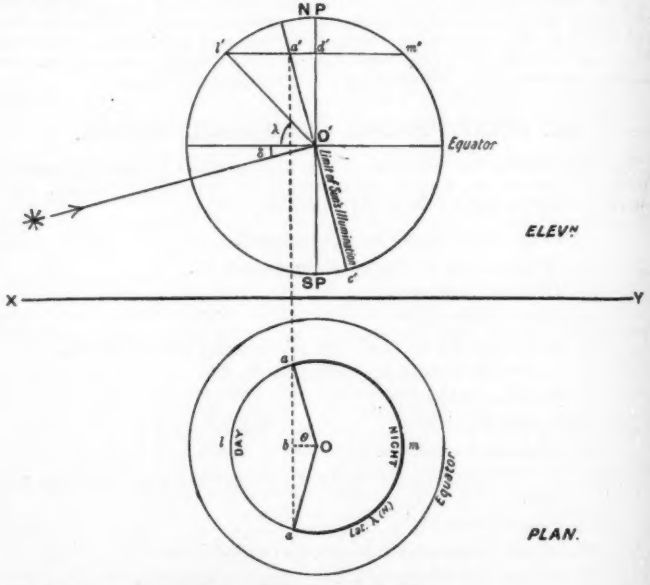
The Committee also recommends that the use of logarithms and algebraical processes should not be forbidden.

75. The Committee recommend that the following be not set in such compulsory examinations:

(1) Rule for H.C.F. by Alternate Division.
(2) Complicated Fractions.
(3) Recurring Decimals.
(4) Mensuration formulae other than those for the triangle, circle, prism, and circular cylinder.
(5) Inverse Compound Interest.
(6) Present Worth and Discount.
(7) Specialised Financial Arithmetic.
(8) Arithmetical Problems which have become standardised, and have little intrinsic value (see 73).
(9) Elaborate and artificial questions of all kinds (see 37, 53, 63, 67).

DURATION OF DAY.

THE following is a simple method of establishing the formula for the determination of the duration of day, at any latitude, for a known declination of the sun.



The figure is an elevation and plan of the earth.

Referring to the elevation, λ is the angle of latitude, δ the angle of declination of the sun on the date considered, $a'O'c'$ the elevation of the

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terminator or limit of the sun's illumination of the earth (at right angles to the line showing the direction of the sun's rays), a' is the point at which the terminator, $a'O'c'$, cuts the elevation of the line of latitude λ.

Referring to the plan, a and a are the points in plan corresponding to a' in elevation; O is the plan of the polar axis.

It is evident that a place on the circle of latitude λ will receive light while moving along the path represented by the arc *ala* (plan), and be in darkness while moving along the path represented by the arc *ama*.

Since these arcs are proportional to the angles they subtend at the centre, we have,

$$\frac{\text{duration of day}}{24 \text{ hours}} = \frac{2\theta}{360^\circ} = \frac{\theta}{180^\circ}.$$

Referring to the figure, it is seen that:

$$a'd' \text{ (elevation)} = bO \text{ (plan)},$$
$$l'd' \text{ („)} = Oa \text{ (plan)},$$
$$\angle O'l'd' \text{ („)} = \lambda,$$
$$\angle a'O'd' \text{ („)} = \delta.$$

Now $O'd' = R \sin \lambda$,

and $l'd' = R \cos \lambda$ (where R is the radius of the earth).

Again, $a'd' = O'd' \tan \delta$

$$= R \sin \lambda \tan \delta.$$

Hence $bO \text{ (plan)} = R \sin \lambda \tan \delta$.

Let $\angle aOa \text{ (plan)} = 2\theta$, then $\angle aOb = \theta$,

and $$\cos \theta = \frac{bO}{Oa} = \frac{R \sin \lambda \tan \delta}{R \cos \lambda}.$$

$$= \tan \lambda \tan \delta.$$

Hence $\theta = \cos^{-1}(\tan \lambda \tan \delta)$, which determines θ.

As before, $$\frac{\text{length of day}}{24 \text{ hours}} = \frac{\theta}{180};$$

$$\therefore \text{ length of day} = \frac{\cos^{-1}(\tan \lambda \tan \delta)}{180} \times 24 \text{ (hrs.)}.$$

The application of the formula to the special cases (i) in which $\delta = 0$, (ii) the determination of λ for which $\theta = 0$ when $\delta = 23\frac{1}{2}^\circ$, (iii) in which $\lambda = 0$, (iv) in which $\lambda > 66\frac{1}{2}^\circ$ and $\delta = 23\frac{1}{2}^\circ$ is interesting.

The formula can be also used for finding the times of sunrise and of sunset.

A. H. Bell.

Sheerness Technical Institute.

Obituary.

Prof. W. H. H. Hudson.

In the late Prof. W. H. H. Hudson, the Association has lost a member of thirty years' standing. He was an earnest and vigorous promoter of many causes in which his ripe judgment, his engaging personality, and profound sympathy will be sorely missed. For some twenty years he was Fellow and Lecturer at St. John's College, Cambridge, and there will be among the members of that society, dating back between the '60's and the '80's, as universal a chorus of appreciation as would be raised by the many pupils with whom he was later associated at King's College, London, and Queen's College, Harley Street. It was

with peculiar satisfaction that one so closely identified with the movement for the Higher Education of Women found himself able to give the benefit of his enthusiasm and his experience to his fellow governors of Newnham. His work upon the Councils of the London Mathematical Society and of the Mathematical Association, of which he was for many years a distinguished and active member, was highly appreciated. Pupils were constantly passing through his hands to take up their posts as teachers of Mathematics, and he was thus enabled indirectly to sow the good seed which bore fruit eventually in the success of the movement for reform in the teaching of this subject. His own teaching was sound and thorough, and based upon principles which, as a writer in a contemporary has remarked, "received remarkable vindication in the record of his family." His only son showed promise of being one of the most remarkable men of his time, and the three daughters who have survived him are also evidence of the combined effect of nature and nurture upon the members of a highly gifted family. His life was not without its tragedies, but he faced the decrees of Fate with signal fortitude, and in this, as in all else, was a noble example to those who were admitted to the privilege of his friendship.

MATHEMATICAL NOTES.

463. [V. 1. a. δ, ϵ.] *On Some Arithmetical Conventions.*

A contention that the value of $7-3\times2$ may be either 1 or 8 recently surprised me, and has led me to look somewhat carefully into the conventions as to the sequence in which arithmetical operations are to be performed.

Many arithmetical books, in their chapter on Fractions, lay down three rules of interpretation. Stated in their baldest form these rules are:

1. Multiplications and divisions must be performed before additions and subtractions. This assigns a meaning to $27\pm5\times3$, say $a\pm b\times c$.
2. Multiplications and divisions must be performed in order (from left to right). This assigns a meaning to $a\div b\times c$, viz. $\frac{ac}{b}$.
3. The word 'of' is, however, equivalent to a bracket. According to this,

$$a\div b \text{ of } c \text{ means } \frac{a}{bc}.$$

It will be convenient to state at once the conclusions I have reached, before entering into the arguments on the subject.

These are, that the Rule 1, though not always happily expressed, is a rule of fundamental importance, and is essential to the harmony of arithmetic and algebra; but that Rules 2 and 3 are of an artificial character, that they are not necessary and that they cannot be defended.

There is little doubt that Rule 1 has suffered from being found in bad company. The case for its separate existence seems to be (apart from mere authority, though Tannery and Workman both adopt it):

(a) Algebraic and arithmetical expressions consist of terms.

(b) Apart from brackets + and − separate terms.

(c) There is no essential distinction between ab, $a.b$ and $a\times b$. Each denotes the product of a and b.

If a and b are numbers expressed in figures, the sign × must be used [for . may be confused with a decimal point, *Enc. des Sc. Math.* I, 1, i, p. 40]. Contrast $2{\cdot}4\times3{\cdot}1$, $2{\cdot}4\,.\,3{\cdot}1$ and $2{\cdot}43{\cdot}1$.

The rule can now be stated in the following way. Apart from brackets, terms must be evaluated before they can be combined by addition or subtraction. A term may contain several factors.

Examples illustrating the necessity of the rule.

(i) Evaluate $a-bc^2$ when $a=10$, $b=3.14$, $c=1{\cdot}4$.

The expression $=10-3.14\times(1{\cdot}4)^2$, etc.

(ii) A square plate, side 10 cm., has a rectangular projection 3·5 by 1·7 cm. What is the total area of the plate and projection?

Area $=100+3{\cdot}5\times1{\cdot}7$, etc.

Observe how the beginner will naturally build up the result term by term.

It will be found on examination of the current text-books that many of them, though they state the Rule 1 in the chapter on Fractions, employ it at a very much earlier stage; *e.g.* in the chapter on Multiplication they may ask pupils to find the value of such expressions as

$$3\times20+17, \quad 18\times112+28, \quad 7\times12+9.$$

[Some apparently would maintain that $3\times20+17$ is clear, but $17+3\times20$ ambiguous—another odd consequence of abandoning Rule 1.]

The objectors to the rule contend that it is always possible to avoid any shadow of doubt as to the meaning of an expression by the free use of brackets:

$$(ax)+(by)+c, \quad (18\times112)+(28\times19), \text{ etc.}$$

To this I reply that few experienced teachers will be found to maintain that piling up brackets makes things easier for a beginner.

Moreover, when the young student comes to algebraic expressions of the form $a-bc$ or $1+xy$, a nasty little trap has been laid for him if he has been led to suppose that $17-3\times4$ or $11+4{\cdot}1\times3{\cdot}2$ are capable of two interpretations.

As regards Rule 2 it is to be noted that it is a peculiarly English rule. The sign of division, ÷, is practically unknown abroad. Thus Tannery, *Leçons d'Arithmétique*, lays down Rule 1 explicitly, but has no Rule 2, for he does not need it. $\frac{ab}{c}$ is shorter and plainer than $a\times b\div c$, and *a fortiori* than $a\div c\times b$.

Rule 3 is still more obviously unnecessary, is equally insular, and seems a mere device for catching the unwary.

I agree in principle that it is an important part of education in mathematics to learn not to be unwary—but the subject is one of considerable extent, and offers other opportunities for the inculcation of care. C. S. JACKSON.

I cordially endorse all that Mr. Jackson says. (1) The recognition of terms in such an expression as $a\times b+c\times d+e\times f$ is absolutely fundamental, and brackets are unnecessary and superfluous. There are so many cases in which brackets are needed that they should be avoided when their presence is not required. (2) Such an expression as $a\div b\times c$ is really ambiguous, because if it means $a\times c\div b$ it should be so written, or be written $(a\div b)\times c$. The first mode of writing it is too much like a deliberate trap. Similarly, if it means $a\div(b\times c)$ it should be so written, unless of course the fractional form $\frac{a}{b\times c}$ is adopted, which, however, when b and c are themselves fractional, has a needlessly complicated appearance.

Brackets should be used to prevent real ambiguities, not to spoon-feed careless pupils.

(3) In some quarters there is apparently a desire to ignore Rule 1, and to extend Rule 2 to a series of numbers connected by the four signs +, −, ×, ÷. This would make a hopeless antagonism between arithmetic and algebra, and would make an arithmetical expression such as $a\times b+c\times d+e\times f$ equivalent to $abdf+cdf+ef$ instead of merely $ab+cd+ef$. No words seem strong enough

to use in repudiating such an interpretation. Even if restricted to × and ÷, Rule 2 is uncomfortably like a trap, and, if extended to + and − as well, it leads to an absurdity. I maintain therefore that the rule that operations should be performed from left to right (which might be called the chain-rule) is erroneous in some cases and a mere trap in others, and should be expunged.

P.S.—May I add that I am sorry to see so great an authority on printing as Professor G. H. Bryan advocating that $n(n+1)(n+2)$ should be printed as $n\,.\,n+1\,.\,n+2$ [see *Gazette*, Jan. 1916, p. 220, § 4]. This is neither the chain-rule nor any other rule, and the saving in expense is negligible. It is a case in which brackets really are needed.

I am glad to see, on the other hand, from § 6 on the same page, that he does not apply the chain-rule to the interpretation of $p+q/2$, or $r/3+s/4$, but obeys the Term Rule.

ALFRED LODGE.

464. [D. 2. d.; V. a. c.] *On the Successive Convergents of a Continued Fraction.*

The following may be of interest to teachers of algebra. In proving the relations between successive convergents of a continued fraction

$$a_1+\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots+\frac{a_n}{b_n}+\cdots$$

it is customary to proceed as follows. We first prove

$$p_1/q_1=a_1/1\,;\quad p_2/q_2=(a_1b_2+a_2)/b_2\,;$$
$$p_3/q_3=(a_1b_2b_3+a_1a_3+a_2b_3)/(b_2b_3+a_3)=(b_3p_2+a_3p_1)/(b_3q_2+a_3q_1).$$

We thus assume

$$p_n=b_np_{n-1}+a_np_{n-2}\,;\quad q_n=b_nq_{n-1}+a_nq_{n-2},$$

and proceed to prove it by induction.

This is rather a big jump for the beginner. In any case, a direct proof is always to be preferred to an induction. The following proof is simple and direct. It is of course assumed that there is no cancelling.

It is obvious that

$$p_n/q_n=(b_nu+a_nv)/(b_nu'+a_nv'),\quad n>1,$$

u, v, u', v' being some numbers. To get p_{n+1}/q_{n+1} we use

$$\frac{a_n}{b_n+}\frac{a_{n+1}}{b_{n+1}},\quad i.e.\ a_nb_{n+1}/(b_nb_{n+1}+a_{n+1}),$$

instead of a_nb_n. Hence

$$p_{n+1}/q_{n+1}=\{(b_nb_{n+1}+a_{n+1})u+a_nb_{n+1}v\}/\{(b_nb_{n+1}+a_{n+1})u'+a_nb_{n+1}v'\}$$
$$=(b_{n+1}p_n+a_{n+1}u)/(b_{n+1}q_n+a_{n+1}u').$$

To get p_{n+2}/q_{n+2}, we change a_{n+1}/b_{n+1} into $a_{n+1}b_{n+2}/(b_{n+1}b_{n+2}+a_{n+2})$.

Hence

$$p_{n+2}/q_{n+2}=\{(b_{n+1}b_{n+2}+a_{n+2})p_n+a_{n+1}b_{n+2}u\}/\{(b_{n+1}b_{n+2}+a_{n+2})q_n+a_{n+1}b_{n+2}u'\}$$
$$=(b_{n+2}p_{n+1}+a_{n+2}p_n)/(b_{n+2}q_{n+1}+a_{n+2}q_n).$$

Thus for $u=3, 4, 5, \ldots$, we have

$$p_n=b_np_{n-1}+a_np_{n-2},$$
$$q_n=b_nq_{n-1}+a_nq_{n-2}.$$

S. BRODETSKY.

465. [V. 1. a. λ.] *A simple method of applying the equation*

$$y=a_0+a_1x+a_2x^2+a_3x^3,\ etc.,$$

to curves, with applications.

Ordinarily where the first, second or higher differentials of curves have to be plotted, at least for the higher derivatives, the only satisfactory

method is one depending on getting sufficiently accurate expressions for the curve and plotting the differential expressions of the equations.

In order to obtain the greatest accuracy as many points should be taken on the curve as possible, and then in the equation

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n,$$

if $n+1$ points are taken, as many as $(n+1)$ simultaneous equations would have to be solved. This ordinarily is a very tedious procedure, even if determinants are being used. However, the tediousness is very much relieved if the device outlined below is followed.

If we take as our first point of reference of the curve the ordinate y_0 corresponding to x_1, and the ordinate y_1 corresponding to the point x_1+p, and thus for the ordinate y_n the coordinate x_1+np, we arrive at the following system of equations:

$$\begin{aligned}
y_0&=a_0+\quad a_1x_1\quad+\quad a_2x_1^2\quad+\ldots+a_nx_1^n,\\
y_1&=a_0+a_1(x_1+p)+a_2(x_1+p)^2+\ldots+a_n(x_1+p)^n,\\
y_2&=a_0+a_1(x_1+2p)+a_2(x_1+2p)^2+\ldots+a_n(x_1+2p)^n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_n&=a_0+a_1(x_1+np)+a_2(x_1+np)^2+\ldots+a_n(x_1+np)^n.
\end{aligned}$$

Instead of attempting to solve these equations in the usual manner, it can be shown that the $(n+1)$th coefficient is given by the following direct simple formula:

$$a_n=\frac{1}{(-p)^n\lfloor n}\left\{y_0-ny_1+\frac{n(n-1)}{\lfloor 2}y_2-\frac{n(n-1)(n-2)}{\lfloor 3}y_3+\ldots+y_n\right\}.$$

Having obtained the $(n+1)$th term, it is now only necessary to arrange a new set of coefficients:

$$\begin{aligned}
y_0-a_nx_1^n&=a_0+a_1x_1+a_2x_1^2+\ldots+a_{n-1}x_1^{n-1}&&=y_{0.1},\\
y_1-a_n(x_1+p)^n&=a_0+a_1(x_1+p)+a_2(x_1+p)^2+\ldots+a_{n-1}(x_1+p)^{n-1}&&=y_{1.1},\\
y_2-a_n(x_1+2p)^n&=a_0+a_1(x_1+2p)+a_2(x_1+2p)^2+\ldots+a_{n-1}(x_1+2p)^{n-1}&&=y_{2.1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_{n-1}-a_n(x_1+np)^n&=a_0+a_1(x_1+np)+a_2(x_1+np)^2+\ldots a_{n-1}(x_1+np)^{n-1}&&=y_{n-1,1},
\end{aligned}$$

which can be solved for a_{n-1} in precisely the same manner as before by adopting the same general formula.

Proof. If we arrange progressively and systematically to subtract from each equation of the system the one immediately following, we arrive at the following scheme of reductions for a set of 5 equations:

$$\begin{array}{lllll}
y_0 & & & & \\
 & y_0-y_1 & & & \\
y_1 & & & & \\
 & y_0-y_1-(y_1-y_2) & & & \\
 & y_1-y_2 & \{y_0-y_1-(y_1-y_2)\}-\{y_1-y_2-(y_2-y_3)\} & & \\
y_2 & & & & \\
 & y_1-y_2-(y_2-y_3) & & [\{y_0-y_1-(y_1-y_2)\}-\{y_1-y_2-(y_2-y_3)\} & \\
 & y_2-y_3 & \{y_1-y_2-(y_2-y_3)\}-\{y_2-y_3-(y_3-y_4)\} & -\{y_1-y_2-(y_2-y_3)\} & \\
y_3 & & & +\{y_2-y_3-(y_3-y_4)\}]=Y & \\
 & y_2-y_3-(y_3-y_4) & & & \\
 & y_3-y_4 & & & \\
y_4 & & & &
\end{array}$$

where $Y=y_0-4y_1+6y_2-4y_3+y_4$.

In fact, if we take a series of $(n+1)$ *equations we obtain analogously the final expression*

$$y_0 - ny_1 + \frac{n(n-1)}{\lfloor 2}y_2 - \frac{n(n-1)(n-2)}{\lfloor 3}y_3 + \ldots + y_n.$$

However, each equation, as the kth equation y_{k-1}, has an nth term with the coefficient a_n. Summing these together, we arrive at the expression

$$a_n\left\{x_1^n - n(x_1+p)^n + \frac{n(n-1)}{\lfloor 2}(x_1+2p)^n + \ldots + (x_1+np)^n\right\}.$$

Now it can easily be shown (or checked) that this latter expression is independent of x_1, and is in fact only dependent on the values p and n.

Thus it can be shown that

$$a_n\left\{x_1^n - n(x_1+p)^n + \frac{n(n-1)}{\lfloor 2}(x_1+2p)^n + \ldots + (x_1+np)^n\right\} = (-p)^n \lfloor n.$$

If now we refer to the general system of reduction, we shall arrive at, for the n *equations* beginning with y_0 and ending with y_{n-1}, or with y_1 and ending with y_n, the same value of

$$a_{n-1}[x_1^{n-1} - (n-1)(x_1+p)^{n-1} + \ldots \{x_1+(n-1)p\}^{n-1}].$$

It therefore follows that, by adopting the above systematic reduction for the $(n+1)$ *equations*, each term involving a coefficient other than a_n will vanish. Hence we have the equation

$$y_0 - ny_1 + \frac{n(n-1)}{\lfloor 2}y_2 - \frac{n(n-1)(n-2)}{\lfloor 3}y_3 + \ldots + y_n$$

$$= a_n\left\{x_1^n - n(x_1+p)^n + \frac{n(n-1)}{\lfloor 2}(x_1+2p)^n + \ldots + (x_1+np)^n\right\} = a_n(-p)^n \lfloor n,$$

and therefore

$$a_n = \frac{1}{(-p)^n \lfloor n}\left\{y_0 - ny_1 + \frac{n(n-1)}{\lfloor 2}y_2 - \frac{n(n-1)(n-2)}{\lfloor 3}y_3 + \ldots + y_n\right\},$$

which is independent of x_1.

Testing the degree of a curve, etc. Since we have the equation

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$

it follows that

$$\frac{d^ny}{dx^n} = a_n \lfloor n,$$

and hence

$$\frac{d^ny}{dx^n} = (-p)^{-n}\left\{y_0 - ny_1 + \frac{n(n-1)}{\lfloor 2}y_2 + \ldots + y_n\right\}.$$

Thus, to test the degree of a curve, taking an arbitrary y_0 with a given value of p, the value of $\frac{d^ny}{dx^n}$ is calculated. Again, if a new value of y is taken, as y_0', the same value of $\frac{d^ny}{dx^n}$ should be arrived at, since the result is independent of any value x_1 or x_1'. This is a check on the constancy of the coefficient a_n.

It follows, moreover, that if the curve is really of the nth degree, then, on the basis of any $(n+2)$ ordinates y, $\frac{d^{n+1}y}{dx^{n+1}}$ should approach zero. Thus, starting with a few terms in the equation for y, and at first taking large values of p, and smaller intermediate values for p when more terms are taken in equation y, by means of varying the initial value x_1, the above furnishes a check on the accuracy of the degree of the curve as well as on that of the inferior derivations.

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466. [K¹. 2. a, b.] If ABC is a triangle whose inscribed and circumscribed circles are fixed, and if S is a fixed point on the latter, then the pedal line of S passes through a fixed point.

From S draw tangents ST, SU to the given inscribed circle, meeting the circle ABC in T, U. Then STU must be another position of ABC.

Let KL be the other line parallel to TU, touching the inscribed circle, and let it meet AB in K, AC in L.

Then AKL, STU are two triangles circumscribed to one circle, and therefore A, K, L, S, T, U lie on a conic. The axis of this conic is equally inclined to AS and TU (because A, S, T, U lie on a circle), and therefore to AS and KL. Hence A, S, K, L lie on a circle, and the feet of the perpendiculars from S to KL, $AK(AB)$ and $AL(AC)$ lie on a line.

Thus the pedal line of S for the triangle ABC passes through the foot of the perpendicular from S to KL. A. C. Dixon.

CORRESPONDENCE.

MULTIPLICATION AND DIVISION OF DECIMALS.

In the Report of the Committee on p. 167 of the *Gazette* for December, the following sums are set out as examples, with the results as follows:

176·3 is multiplied by 3·41, and the product is stated to be 601·183.

1·763 is divided by 3·41, and the quotient stated to be ·517... with a remainder 2390.

Neither of these results is correct, if decimals have the meaning which is implied in all physical and the great majority of mathematical calculations. Under these conditions 601·183 represents the product, not of 176·3 and 3·41, but of 176·300 and 3·41000. By 176·3 we mean a decimal not less than 176·25 and not greater than 176·349, and similarly for the other factor. But if 176·25 be multiplied by 3·405, the product to three places of decimals appears to be 600·131, and if 176·35 is multiplied by 3·415, the product appears to be 602·235, if my calculations are correct. Even if there is a slip in my work, 601·183 cannot possibly represent the product of these two decimals correctly to three places unless it is correct to write the factors with the zeros inserted, as has been done above.

Again, the division sum represents the division, not of 1·763 by 3·41, but of 1·76300 by 3·4100. It is only necessary to divide 1·7635 by 3·405, or 1·7625 by 3·415, to show that, avoiding the common accepted meaning of these decimals, the figures brought down in the second and third lines are incorrect.

In most applications multiplication of exact decimals is an operation which does not occur once in a thousand times, and it is therefore quite unnecessary to teach pupils a method which is liable to lead to incorrect results in the remaining 999 times. It is much better that they should begin by adding zeros when the decimals are exact, and that they should be told *never to extend the operation to the right of the first line.*

In reply to these criticisms it will doubtless be objected that contracted methods have been abandoned by many schoolmasters on the ground that they consider that their pupils find them difficult to learn. May I point out, however, that it is not the methods here advocated, but the methods commonly taught, which are contracted. The trouble arises entirely out of the fact that the pupils are previously taught to

use contracted methods in the multiplication and division of *integers*. The following examples will make it clear :

Long Method.		Usual Method.	
1763		1763	
341		341	
528900		5289	
70520		7052	
1763		1763	
601183		601183	
341) 601183	(1763	341) 601183	(1763
341000		341	
260183		2601	
238700		2387	
21483		2148	
20460		2046	
1023		1023	
1023		1023	

Which are the contracted methods ? Obviously the second. (In dealing with multiplication of integers, the multiplications are often performed in the reverse order, but this does not affect the issue.)

If the pupils are taught to perform their multiplications and divisions in full when operating on integers, as a preliminary to the study of decimals, instead of using these contracted methods, they should have no difficulty in applying the methods to calculations with decimals suitable for dealing with approximations.

The fact that 70 per cent. of the Public Schools have declared in favour of the proposals contained in the circular does not affect the more important question as to what use their pupils are likely to make of the teaching after they have left school. In my experience as an examiner I am constantly coming across pages and pages of multiplications involving strings of twenty or more figures, and leading to an answer of which even the first significant digit is wrong and the decimal point hopelessly out of place. These confused masses of figures appear in the work of candidates even after they have spent some years at a university. It is surely penny wise and pound foolish to begin children on a method which may be shorter at first, but which leads to all this useless waste of effort later on. G. H. Bryan.

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